1. *Let (X, d) be a metric space. Let $C_b(X)$ denote the normed linear space consisting of all (real or complex valued) bounded functions on X, with usual operations and supremum norm.*

   (a) *Show that $C_b(X)$ is a Banach space.*

   (b) *Fix a point $x_0 \in X$. For any $x \in X$, let $\rho_x : X \to \mathbb{R}$ be defined by $\rho_x(y) = d(x, y) - d(x_0, y), y \in X$. Show that $x \to \rho_x$ is an isometric embedding of X in $C_b(X)$.*

   (c) *show that every metric space occurs as a dense subspace of a complete metric space.*

Solution: □

(a) We show that $C_b(X)$ is a Banach space. Let k denote the field $\mathbb{R}$ or $\mathbb{C}$. Let (f_n) be a Cauchy sequence in $C_b(X)$. Then note for each $x \in X, |f_n(x) - f_m(x)| \leq \|f_n - f_m\| \to 0$ as $m, n \to \infty$ so that $(f_n(x))$ is a Cauchy sequence in k for each $x \in X$ and hence convergent in k. Let $f : X \to k$ be defined by $f(x) = lim f_n(x), x \in X$. We need to show that f is in $C_b(X)$. Given $\epsilon > 0$, there exists a positive integer N such that $\|f_n - f_m\| < \epsilon$ for all $m, n \geq N$. Thus, $|f_n(x) - f_m(x)| < \epsilon$ for all x in X and for all $m, n \geq N$. Let $m \to \infty$. Then,

$$|f_n(x) - f(x)| \leq \epsilon, \forall x \in X, \forall n \geq N. \tag{1}$$

Thus for all $x \in X$,

$$\begin{aligned} |f(x)| =& |f(x) - f_N(x) + f_N(x)| \\ & \leq |f(x) - f_N(x)| + |f_N(x)| \\ & \leq \epsilon + \|f_N\| \end{aligned}$$

showing that f is bounded on X. Thus we conclude that f belongs to $C_b(X)$. Also it is clear from the equation (1) that $f_n \to f$ in $C_b(X)$. Therefore $C_b(X)$ is a Banach space.

(b) Given $x, y \in X$, by an appeal to the triangle inequality in (X, d) we see that $|d(x,z) - d(y,z)| \leq d(x,y)$ for all $z \in X$ and thus $\|\rho_x - \rho_y\| = Sup_{z \in X}|d(x,z) - d(y,z)| \leq d(x,y)$. With $z = y$, we observe that $d(x,y) \leq \|\rho_x - \rho_y\|$. Thus, $\|\rho_x - \rho_y\| = d(x,y)$.

(c) Given any metric space (X, d), by virtue of previous parts (a) and (b) of the problem we see that there is an isometric embedding of X in $C_b(X)$ given by $x \to \rho_x$. Let A denote the closure of $\{\rho_x : x \in X\}$ in $C_b(X)$. Then (A, d) is a complete metric space (being closed subspace of a complete metric space) of which X is a dense subspace.

2. Let$X$ be a complex Banach space. Let $X_{\mathbb{R}}$ denote the same space, viewed as a real Banach space. Show that $f \to Re(f)$ is an isometry from X^* onto $X_{\mathbb{R}}^*$.

Solution: Given $f \in X^*$, we note that for any $x \in X, f(x) = |f(x)|e^{i\theta}$ where $\theta \in [0, 2\pi)$. Therefore, $|f(x)| = f(x)e^{-i\theta} = f(e^{-i\theta}x) = Ref(e^{-i\theta}x)$. So, $|f(x)| = Ref(e^{-i\theta}x) \leq \|Ref\|\|x\|$ so that$\|f\| \leq \|Ref\|$. On the other hand it is obvious that $\|Ref\| \leq \|f\|$ and the desired equality follows. $\square$

3. (a) Prove that every non-empty closed and convex subset of a Hilbert space has a unique element of smallest norm.

(b) Let C be the Banach space of all continuous function on $[0, 1]$ into $\mathbb{C}$, with supremum norm. Let $M = \{f \in C : \int_0^{\frac{1}{2}} f(t)dt - \int_{\frac{1}{2}}^1 f(t)dt = 1\}$. Show that M is a closed and convex non-empty subset of C containing no element of smallest norm.

Solution:

(a)

Suppose K is a non-empty, closed and convex subset of a Hilbert space. Let

$$\delta = \inf \{\|x\| : x \in K\}.$$

Let $x, y \in K$. Then $\frac{1}{2}(x + y) \in K$ and it follows from the parallelogram law that

$$\|x - y\|^2 = 2(\|x\|^2 + \|y\|^2) - \|x + y\|^2 \leq 2(\|x\|^2 + \|y\|^2) - 4\delta^2 \qquad (2)$$

Choose a sequence (z_n) in K such that $\|z_n\|$ converges to δ. Now an appeal to the equation (2) shows that

$$\|z_n - z_m\|^2 \leq 2(\|z_n\|^2 + \|z_m\|^2) - 4\delta^2 \to 0$$

as $m, n \to \infty$ so that (z_n) is a Cauchy sequence in K and as K is closed, (z_n) converges to some point of K, say, z. Then $\|z\| = lim\|z_n\| = \delta$. Hence the existence of an element of smallest norm in K is ensured.

If $z_1, z_2 \in K$ satisfy $\|z_1\| = \|z_2\| = \delta$, then it follows from the equation (2) that $\|z_1 - z_2\|^2 \leq 0$ so that $z_1 = z_2$ and the uniqueness follows.

(b) It is clear that M is closed and eqconvex non-empty subset of C. Let $f \in M$. Then

$$1 = |\int_0^{\frac{1}{2}} f(t)dt - \int_{\frac{1}{2}}^1 f(t)dt| \leq \|f\|$$

.

Thus $\inf\{\|f\| : f \in M\} \geq 1$. For each $n \geq 2$, let f_n denote the continuous function on $[0,1]$ whose graph is the union of line segments from $(0,1)$ to $(\frac{1}{2}, 1)$, then from $(\frac{1}{2}, 1)$ to $(\frac{1}{2} + \frac{1}{n}, \frac{1+n}{1-n})$ and finally from $(\frac{1}{2} + \frac{1}{n}, \frac{1+n}{1-n})$ to $(1, \dfrac{1+n}{1-n})$. Then one can see that

$$\begin{aligned}
\int_0^{\frac{1}{2}} f(t)dt - \int_{\frac{1}{2}}^1 f(t)dt &= \int_0^{\frac{1}{2}} f(t)dt - \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} f(t)dt - \int_{\frac{1}{2}+\frac{1}{n}}^1 f(t)dt \\
&= \frac{1}{2} - \frac{1}{n(1-n)} - \frac{(1+n)(n-2)}{2n(1-n)} \\
&= 1
\end{aligned}$$

so that $f_n \in K$ for all $n \geq 2$ and one can easily see that $\|f_n\| = \frac{n+1}{n-1}$ and so, $\inf\{\|f_n\| : n \geq 2\} = 1$. Consequently, $\inf\{f : f \in M\} \leq 1$ whence $\inf\{f : f \in M\} = 1$. We assert that there is no element f in M such that $\|f\| = 1$. Suppose there is such an f. Then writing $f = u + iv$ we see that

$$1 = \int_0^{\frac{1}{2}} u(t)dt - \int_{\frac{1}{2}}^1 u(t)dt$$

which implies that

$$\int_0^{\frac{1}{2}} (u(t) - 1)dt + \int_{\frac{1}{2}}^1 (-u(t) - 1)dt = 0. \tag{3}$$

Now note that $\|f\| = 1$ tells that $|u| \leq 1$ and hence, both $u - 1$ and $-u - 1$ are non-positive integrands and it the immediately follows from the equation (3) that

$u = 1$ on $[0, \frac{1}{2})$ and $u = -1$ on $(\frac{1}{2}, 1]$ so that u is discontinuous at $\frac{1}{2}$, a contradiction. Hence our assertion is established.

□

4. Let $K : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be defined by $K(x, y) = min(x, y)$.

   (a) Prove that K is an n.n.d. kernel. Let H denote the Hilbert space with reproducing kernel K.

   (b) Show that every element f of H is a continuous function with $f(0) = 0$.

   (c) Let $0 = x_0 < x_1 < x_2 < \cdots < x_n$ and $m_1, m_2, \cdots, m_n$ be real numbers. Let f be the unique continuous function on $[0, \infty)$ such that $f(0) = 0, f(x) =$ constant for $x > x_n$, and $f|_{[x_i 1, x_i]}$ is a linear function of slope $m_i, 1 \leq i \leq n$. Show that $f \in H$ and compute its norm.

Solution:

(a) Consider the function $\phi : [0, \infty) \to L^2([0, \infty))$ given by $\phi(x) = 1_{[0,x]}$ (characteristic function on $[0, x]$). Then it is obvious that $\langle \phi(x), \phi(y) \rangle = \int 1_{[0,x]} 1_{[0,y]} d\mu = K(x, y)$, showing that K is a positive definite kernel.

(b) Let $\mathcal{H}$ denote the Hilbert space with reproducing kernel K. Let $\mathcal{H}_0$ denote the subspace of $\mathcal{H}$ spanned by the functions $\{K(., x) : x \in [0, \infty)\}$. It is known that $\mathcal{H}_0$ is dense in $\mathcal{H}$. Let $f \in \mathcal{H}$. Then there is a sequence (f_n) in $\mathcal{H}_0$ such that $f_n \to f$. Then f_n converges pointwise to f. So, $f_n(0) \to f(0)$. Evidently $g(0) = 0$ for any $g \in \mathcal{H}_0$ so that $f(0) = 0$.

   We now show that f is continuous. We use the notation e_x to denote the evaluation functional for any x in $[0, \infty)$. Then e_x are continuous linear functionals on $\mathcal{H}$ with $\|e_x\| = K(x, x)^{\frac{1}{2}} = x^{\frac{1}{2}}$. First note that given any $x \in [0, \infty), K(., x)$ is the function which is identity function on $[0, x]$ and on $[x, \infty)$, it is the constant function x and hence $K(., x)$ is continuous. Thus every element of $\mathcal{H}_0$ is continuous (being linear combination of continuous functions). Given $\epsilon > 0$ and $x_0 \in [0, \infty)$. Since $f_n \to f$, there exists a positive integer N such that $\|f_n - f\| < \frac{\epsilon}{3(x_0+1)^{\frac{1}{2}}}, \forall n \geq N$. Now continuity of f_N at x_0 suggests that there is a $\delta' > 0$ such that $|f_N(x) - f_N(x_0)| < \frac{\epsilon}{3}$ whenever $|x - x_0| < \delta'$. Let $\delta =$

$\min\{\delta', 1\}$. Thus whenever $|x - x_0| < \delta$, we have that

$$\begin{aligned}
|f(x) - f(x_0)| =& |f(x) - f_N(x) + f_N(x) + f_N(x_0) - f_N(x_0) - f(x_0)| \\
&\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\
&\leq \|e_x\| \|f_N - f\| + |f_N(x) - f_N(x_0)| + \|e_{x_0}\| \|f_N - f\| \\
&= x^{\frac{1}{2}} \|f_N - f\| + |f_N(x) - f_N(x_0)| + x_0^{\frac{1}{2}} \|f_N - f\| \\
&< (x_0 + \delta)^{\frac{1}{2}} \frac{\epsilon}{3(x_0+1)^{\frac{1}{2}}} + |f_N(x) - f_N(x_0)| + (x_0 + \delta)^{\frac{1}{2}} \frac{\epsilon}{3(x_0+1)^{\frac{1}{2}}} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,
\end{aligned}$$

proving that f is continuous.

(c) A little thought shows that

$$f = (m_1 - m_2)K(., x_1) + (m_2 - m_3)K(., x_2) + \cdots + (m_{n-1} - m_n)K(., x_{n-1}) + m_n K(., x_n)$$

so that $f \in \mathcal{H}_0 \subset \mathcal{H}$ and

$$\begin{aligned}
\|f\|^2 =& \sum_{i,j=1}^{n-1} (m_i - m_{i+1})(m_j - m_{j+1}) K(x_j, x_i) + m_n^2 K(x_n, x_n) \\
&+ m_n \sum_{i=1}^{n-1} (m_i - m_{i+1}) K(x_n, x_i) + m_n \sum_{i=1}^{n-1} (m_i - m_{i+1}) K(x_i, x_n) \\
&= \sum_{i,j=1}^{n-1} (m_i - m_{i+1})(m_j - m_{j+1}) \min(x_j, x_i) + m_n^2 x_n + 2m_n \sum_{i=1}^{n-1} (m_i - m_{i+1}) x_i \\
&= \sum_{i=1}^{n-1} x_i (m_i^2 - m_{i+1}^2) + m_n^2 x_n.
\end{aligned}$$

□

5. Let $U : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ ($\mathbb{T}$ = unit circle with normalised arc-length measure) be defined by $(Uf)(z) = zf(z), z \in T, f \in L^2(\mathbb{T})$. Prove that U is a unitary and compute its spectrum.

Solution: Note that for any $f \in L^2(\mathbb{T}), \|U(f)\|^2 = \int_{\mathbb{T}} |z|^2 |f(z)|^2 dz = \|f\|^2$, proving that U is unitary. Hence, $\sigma(U)$(Spectrum of U)$\subseteq \mathbb{T}$. Let $\lambda \in \mathbb{T}$. We assert that $\lambda \in \sigma(U)$. Let c be any non-zero complex number and let 1_c denote the constant function c on $\mathbb{T}$. Obviously $1_c \in L^2(\mathbb{T})$. If there exists $f \in L^2(\mathbb{T})$ such that $(U - \lambda I)(f) = 1_c$, then $(z - \lambda)f(z) = c$, for all z outside a set of measure zero, which in turn implies that $f(z) = \frac{c}{z-\lambda}$ for all z outside a set of measure zero. But such an f certainly is not a member of $L^2(\mathbb{T})$. Hence $U - \lambda I$ is not surjective for any $\lambda \in \mathbb{T}$ and consequently our assertion is established. Thus $\sigma(U) = \mathbb{T}$. $\square$